International Journal of Theoretical Physics, Vol. 18, No. 1 (1979), pp. 67–75

Rotating Frames in Special Relativity. II. Generalized Theory

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Received: January 25, 1977

The transformation theory for rotating frames presented in a previous paper (Strauss, 1974) is generalized by replacing the usual condition $\mathbf{r} = \mathbf{R}$ for $\omega\mathbf{R} < \mathbf{c}$ (invariance of radius) by $\mathbf{r} = \mathbf{R}\mathbf{g}(\beta_\mathbf{R})$ so that $\mathbf{r}$ is now defined for all values of $\mathbf{R}$, $\mathbf{0} \leqslant \mathbf{R} \leqslant \infty$. This generalization does not affect the kinematic transformation $\{\boldsymbol{\theta}, \mathbf{T}\} \to \{\vartheta^{(\mathbf{r})}, \{^{(\mathbf{r})}\}$ and the result group structure required by the theoretical constraints previously established, provided the old parameter "$\mathbf{r}$" ($=\mathbf{R}$) is now identified throughout with either $\mathbf{r}$ or $\mathbf{R}$; for physical reasons it must be identified with $\mathbf{R}$. The function $\mathbf{g}$, which cannot be fixed by theoretical constraints, determines the degree of geometrical anisotropy in the rotating plane $\mathbf{z}$ = const. More specifically, since $\mathbf{g}$ enters the expression for the ratio C/D (circumference/diameter) its choice corresponds to the choice of a congruence definition for lengths in radial and tangential directions. While on this (purely geometrical) level $\mathbf{g}$ remains undetermined, it can be uniquely determined experimentally on the kinematic level, e.g., by observing in Σ^ω the motion of a free particle. Thus the supremacy of kinematics over geometry is explicated by a further instance. At the same time, special relativity theory (SRT) is shown to belong to the class of theories with theoretically unsolvable problems.

1. INTRODUCTION

In a previous paper (Strauss, 1974), hereafter referred to as I, a solution of the transformation problem $\Sigma^0 \to \Sigma^\omega$ (inertial frame $\to$ uniformly rotating frame) was presented which satisfies all theoretical constraints there established and which is unique under the usual conditions $\mathbf{z} = \mathbf{Z}$ and $\mathbf{r} = \mathbf{R}$ (invariance of radius), $\mathbf{Z}$, $\mathbf{R}$, $\boldsymbol{\theta}$ being the cylindrical coordinates in Σ^0 with $\mathbf{Z}$ as axis of rotation. In the present paper the condition $\mathbf{z} = \mathbf{Z}$ will be maintained since there appears to be no reason why it should be given up, but the invariance of the radius will not be demanded; instead, we shall consider the

0020-7748/79/0100-0067$03.00/0

general case

$$\mathbf{r} = \mathbf{R}\mathbf{g}(\beta_{\mathbf{R}}) \qquad [\beta_{\mathbf{R}} =_{df} \omega\mathbf{R}/\mathbf{c}] \tag{1.1}$$

where $\mathbf{g}$ is an unspecified function which of course must satisfy

$$g(0) = 1. \tag{1.2}$$

The main reason for considering the general case (1.1) is that it provides a more general basis for discussing questions of intrinsic geometry in the rotating plane: since the latter will depend on $\mathbf{g}$, different choices of $\mathbf{g}$ will correspond to different conventions (operational or coordinative definitions) and/or to different theoretical constraints going beyond those already established in I. This will also make it possible to consider some results of other authors to correspond to a particular choice of $\mathbf{g}$.

Another more specific and more technical reason for considering case (1.1) is as follows. For $\mathbf{r} = \mathbf{R}$ to be consistent with special relativity theory (SRT), the range of $\mathbf{r}$ has to be restricted to values less than $|c/\omega|$, i.e., $\mathbf{r} = \mathbf{R}$ really means

$$\begin{array}{ll} \mathbf{r} = \mathbf{R} & \text{for } |\omega\mathbf{R}| < c \\ \mathbf{r} \text{ undefined} & \text{for } |\omega\mathbf{R}| \geqslant \mathbf{c} \end{array}$$

Such a definitional restriction of a nonperiodic variable appears to be rather odd and objectionable. Instead, one should expect

$$\begin{array}{ll} \mathbf{r} = \text{finite} & \text{for } 0 < |\mathbf{R}\omega| < \mathbf{c} \\ \mathbf{r} = 0 \text{ or } \infty & \text{for } |\mathbf{R}\omega| = \mathbf{c} \\ \mathbf{r} = \text{imaginary} & \text{for } |\mathbf{R}\omega| > \mathbf{c} \end{array}$$

which, together with (1.2), would give the following constraints for the function $\mathbf{g}$:

$$\begin{array}{ll} \text{(i) } \mathbf{g}(0) = 1 & \text{(ii) } \mathbf{g}(1) = 0 \text{ or } \infty \\ \text{(iii) } \mathbf{g}(\mathbf{x}) = \text{imaginary for } |\mathbf{x}| > 1 & \text{(iv) } \mathbf{g}(-\mathbf{x}) = \mathbf{g}(\mathbf{x}) \end{array} \tag{1.3}$$

In addition one should demand that $\mathbf{g}$ be a monotonous function in the interval $0 \leqslant |\mathbf{x}| \leqslant 1$. [The simplest functions satisfying these conditions appear to be $\mathbf{g}_{\pm}(x) = (1 - x^2)^{\pm 1/2}$.]

As mentioned above, the function $\mathbf{g}$ codetermines the intrinsic geometry in the rotating plane, or rather what is left of it (geodesics and the ratio $\mathbf{C}/\mathbf{D}$). (A complete intrinsic geometry in the mathematical sense does not exist in a rotating plane, as shown in I.) This applies in particular to the ratio $\mathbf{C}/\mathbf{D}$ (circumference/diameter, in intrinsic measure), which can be taken as a measure of *geometrical anisotropy* in the rotating plane since it compares a length in tangential direction with a length in radial direction and thus involves an implicit definition of congruence for lengths in these different directions.

If we define "geometrical anisotropy in the rotating plane," henceforth denoted by α, in such a way that it is zero for the Euclidean plane, we can put

$$\alpha =_{df} \mathbf{C}/\pi\mathbf{D} - 1 \tag{1.4}$$

We then have $\alpha = \kappa - 1$ in I and $\alpha = \kappa/\mathbf{g} - 1$ in the generalized theory. In view of existing misinterpretations it should be pointed out that geometrical anisotropy does not imply intrinsic curvature: the latter is given by the appropriate curvature tensor. In our theory it is zero independently of $\mathbf{g}$ everywhere on the surface of a rotating disk, as it ought to be.

2. THE GENERALIZED THEORY

2.1. The Identification of the Old Parameter *r*. An examination of the two-dimensional transformation theory presented in I shows that it is not affected by the generalization (1.1) provided the radial parameter r occurring in the transformation formulas is identified throughout with either $\mathbf{R}$ or $\mathbf{r} = \mathbf{g}(\beta_R)\mathbf{R}$. Thus from the purely formal point of view there exist *two* possible generalizations of the theory presented in I, both having the same group structure. The decision must therefore be based on physical arguments.

There are two compelling physical reasons for identifying the old parameter "r" with $\mathbf{R}$ rather than with $\mathbf{r}$. The first of these reasons follows from the theoretical constraint $C2$ in I concerning the local time metric in the rotating frame in relation to the interial time metric as given by the expression

$$\left(\frac{dt^{(r)}}{dT}\right)_{dr = d\vartheta^{(r)} = 0}$$

As the local time metric cannot depend on the geometrical function $\mathbf{g}$, we have to equate the above expression to κ_R^{-1} (and not to κ_r^{-1}). Besides, it is this value which is experimentally established in experiments with high-energy particles in circular accelerators. (From a more general view, this is an instance of, or an argument for, the priority of chronometry $C2$ from I now reads

$$\left(\frac{dt^{(r)}}{dT}\right)_{dr = d\vartheta^{(r)} = 0} = \kappa_R^{-1} \tag{2.1}$$

with

$$\kappa_R =_{df} [1 - \beta_R^2]^{-1/2} \tag{2.2}$$

$$\beta_R =_{df} \omega R/c \tag{2.3}$$

The second reason for identifying the old parameter "r" with $\mathbf{R}$ rather than with $\mathbf{r} = \mathbf{gR}$ is the requirement of time orthogonality which is a necessary condition for the spatial coordinates to have geometrical meaning. Since

$$dS^2 = dZ^2 + dR^2 + \left[R^2\left(\frac{\partial\theta}{\partial\vartheta}\right)^2 - c^2\left(\frac{\partial T}{\partial\vartheta}\right)^2\right] d\vartheta^2 - \left[\left(\frac{\partial T}{\partial t}\right)^2 - R^2\left(\frac{\partial\theta}{\partial t}\right)^2\right] c^2\, dt^2 + 2\left[R^2 \frac{\partial\theta}{\partial\vartheta}\frac{\partial\theta}{\partial t} - c^2 \frac{\partial T}{\partial t}\frac{\partial T}{\partial\vartheta}\right] s\vartheta\, dt \tag{2.4}$$

the condition for time orthogonality reads

$$R^2/c^2 = \frac{\partial T}{\partial t}\frac{\partial T}{\partial\vartheta}\bigg/\frac{\partial\theta}{\partial\vartheta}\frac{\partial\theta}{\partial t} = \frac{\partial T}{\delta\vartheta}\bigg/\frac{\partial\theta}{\partial t} = \text{“}r\text{”}^2/c^2 \tag{2.5}$$

the last two equations following from I, Section 2.16. [The superscript (r) has been omitted.]

Thus the solution of the transformation problem $\Sigma^0 \rightarrow \Sigma_r^\omega$ (inertial frame $\rightarrow$ cylindrical subframe rotating with uniform angular speed ω with respect to Σ^0) now reads

$$\begin{aligned} z &= Z \\ r &= g(\beta_R)R \\ \vartheta^{(r)} &= \kappa_R[\theta - \omega T] \\ t^{(r)} &= \kappa_R[-(\beta_R{}^2/\omega)\theta + T] \end{aligned} \tag{2.6}$$

or, using the inverse,

$$\begin{aligned} Z &= z \\ R &= G(\beta_r)r \\ \theta &= \kappa_R[\vartheta^{(r)} + \omega t^{(r)}] \\ T &= \kappa_R[(\beta_R{}^2/\omega)\vartheta^{(r)} + t^{(r)}] \end{aligned} \tag{2.7}$$

where the second line is the solution of the second equation in (2.6) for **R**, or

$$G(\beta_r) = g(\beta_R)^{-1} \tag{2.8}$$

Hence the four-dimensional line element is given by

$$dS^2 = dz^2 + (G + \beta_r\dot{G})^2\, dr^2 + G^2r^2\, d\vartheta^{(r)2} - c^2\, dt^{(r)2} \tag{2.9}$$

where $\dot{\mathbf{G}}$ is the derivative of **G** with respect to its argument $\beta_\mathbf{r} = \mathbf{r}\omega/\mathbf{c}$.

Omitting the first line in (2.6) we may write the transformation $\Sigma_r^\omega \leftarrow \Sigma^0$ in the form

$$\{r, \vartheta^{(r)}, t^{(r)}\} = \mathscr{M}_R^{AO}\{R, \theta, T\} \tag{2.10}$$

where rows have been written instead of columns and where

$$\mathscr{M}_R^{AO} =_{df} \begin{pmatrix} g_R^{AO} & 0 \\ 0 & \mathscr{R}_R^{AO} \end{pmatrix} \tag{2.11}$$

with

$$g^{A}_{RO} =_{df} g(\beta^{A}_{OR}) = g(\beta^{O}_{AR}) \qquad \beta^{O}_{AR} =_{df} \omega_A{}^O R/c \tag{2.12}$$

and

$$\mathscr{R}^{AO}_{R} =_{df} \kappa^{O}_{AR}\begin{Bmatrix} 1 & \omega_O{}^A \\ (\beta^{O}_{AR})^2/\omega_O{}^A & 1 \end{Bmatrix} \tag{2.13}$$

Similarly the inverse transformation (2.7) may be written in the form

$$\{R, \theta, T\} = \mathscr{M}^{OA}_{r}\{r, \vartheta^{(r)}, t^{(r)}\} \tag{2.14}$$

where

$$\mathscr{M}^{OA}_{r} =_{df} \begin{pmatrix} G^{OA}_{r} & 0 \\ 0 & \mathscr{R}^{OA}_{R} \end{pmatrix} \tag{2.15}$$

with

$$G^{OA}_{r} =_{df} G(\beta^{O}_{Ar}) \qquad \beta^{O}_{Ar} =_{df} \omega_A{}^O r/c$$

and

$$\mathscr{R}^{OA}_{R} =_{df} \kappa^{O}_{AR}\begin{pmatrix} 1 & \omega_A{}^O \\ \beta^{O2}_{AR}/\omega_A{}^O & 1 \end{pmatrix} = \mathscr{R}(\beta^{O}_{AR}, \omega_A{}^O) \tag{2.17}$$

In these formulas the rotating frame Σ^A is characterized by its angular velocity $\omega_A{}^O$ with respect to the fundamental frame Σ^0, while its cylindrical subframe $\Sigma_r{}^A$ is characterized by the subscript **r** or the corresponding **R**, this notation being the same as in I.

2.2. The General Uniform Rotation Equivalence. The transformation connecting two uniformly rotating subframes $\Sigma^{A}_{r_A}$ and $\Sigma^{B}_{r_B}$ is given by the matrix operator

$$\mathscr{M}^{AB}_{r_A r_B} = \mathscr{M}^{AO}_{r_A}\mathscr{M}^{OB}_{r_B} = \begin{pmatrix} g^{AO}_{R_A} G^{OB}_{r_B} \dfrac{R_A}{R_B} & 0 \\ 0 & \mathscr{R}^{AB}_{R_A R_B} \end{pmatrix} \tag{2.18}$$

from which it is easily shown that these transformations form a group (cf. I, Section 4).

The angular velocity of Σ^A with respect to the subframe $\Sigma^{B}_{r_B}$ works out to be

$$\omega_A{}^B = \frac{\omega_B - \omega_A}{1 - \omega_A\omega_B R_B{}^2 c^{-2}} \tag{2.19}$$

which corresponds to I, equation (4.4). Hence

$$\omega_A{}^B \neq -\omega_B{}^A \qquad \text{unless} \qquad R_A = R_B \tag{2.20}$$

2.3. The Rotating Cylinder Subgroup. The condition $\mathbf{R}_A = \mathbf{R}_B = \cdots = \mathbf{R}$ defines the rotating cylinder subgroup and the corresponding equivalence class of subframes. The transformation formulas are the same as in I, with "r" replaced by **R**. The relation between the intrinsic radii reduces to

$$r_A = [g(\beta_A)/g(\beta_B)]r_B \tag{2.21}$$

with

$$\beta_X =_{df} \omega_X R/c$$

or, equivalently

$$r_A = [G(\hat{\beta}_B)/G(\hat{\beta}_A)]r_B \tag{2.22}$$

with

$$\hat{\beta}_X =_{df} \omega_X r_X/c \tag{2.23}$$

Note that equation (2.22) does not contain **R**; but it still contains a reference to the fundamental frame Σ^0 unless the bracketed expression turns out to be a function of the relative angular velocity $\omega_B{}^A$. This reference to Σ^0 distinguishes the rotating cylinder equivalence of the present theory from that of the former, but it does not prevent the subframe $\Sigma_R{}^0$ from belonging to this equivalence class. Yet it does underline once more the distinguished role of the fundamental (inertial) frames in special relativity. Note that here and in the following the superscript ^{0} referring to Σ^0 is dropped.

2.4. The Rotating Disk Subgroup. This subgroup is defined by the condition $\omega_A = \omega_B = \cdots = \omega$. Different intrinsic radii will again be distinguished by different numerical subscripts, as in I. The transformation formulas are again the same as in I, with "r" replaced by **R**. In particular, we have the invariant

$$\kappa_1^{-1}\vartheta^{(1)} = \kappa_2^{-1}\vartheta^{(2)} \equiv \vartheta \tag{2.24}$$

with

$$\kappa_i^{-1} = [1 - (\omega R_i/c)^2]^{1/2}$$

The relation between the intrinsic radii is given by

$$r_1 = [g(\beta_1)/g(\beta_2)][R_1/R_2]r_2 \tag{2.25}$$

with

$$\beta_i =_{df} \omega R_i/c \tag{2.26}$$

or, equivalently, by

$$r_1 = [G(\hat{\beta}_2)/G(\hat{\beta}_1)[R_1/R_2]r_2 \tag{2.27}$$

with

$$\hat{\beta}_i =_{df} \omega r_i/c \tag{2.28}$$

As these formulas follow from the definitions they are mathematical identities rather than transformation formulas. In particular, they do not enable an observer on the rotating disk to determine its angular velocity with respect to Σ^0, as formula (2.24) does.

2.5. The Geometry on the Rotating Disk. On first sight, one may expect that the noninvariance of the radius essentially affects the geometry on the rotating disk. Now a global geometry on the rotating disk does not exist, as pointed out in I. We may, however, consider the two geometrical remnants already considered in I, viz., the geodesics and the ratio **C/D** where **C** and **D** are the intrinsic measures of circumference and diameter, respectively.

As to the geodesics, we have to start from the expression for the four-dimensional interval (2.9), i.e.,

$$dS^2 = dz^2 + [G(\hat{\beta}) + rG_r]^2\,dr^2 + (rG)^2\,d\vartheta^{(r)2} - c^2\,dt^2 \qquad (2.29)$$

which yields

$$d\sigma^2 = [G + rG_r]^2\,dr^2 + [rG]^2\,d\vartheta^{(r)2} \qquad (2.30)$$

Here, $\mathbf{G}_r$ is the partial derivative of **G** with respect to **r**.

Proceeding in the same way as in I we obtain

$$\dot{\vartheta}^{(r)} = \alpha G^{-2} r^{-2} \qquad (2.31)$$

and

$$\dot{r} = (G + rG_r)^{-1}[1 - \alpha^2 G^{-2} r^{-2}]^{1/2} \qquad (2.32)$$

Hence

$$d\vartheta^{(r)}/dr \equiv \dot{\vartheta}^{(r)}/\dot{r} = \alpha^{-1} X^2 [1 - X^2]^{-1/2}(G + rG_r) \qquad (2.33)$$

or

$$d\vartheta^{(r)} = -[1 - X^2]^{-1/2}\,dX \qquad (2.34)$$

with

$$X =_{df} \alpha G^{-1} r^{-1} \qquad (2.35)$$

Hence, by integration,

$$\vartheta^{(r)} = \vartheta_0^{(r)} + \arcsin(-\alpha G^{-1} r^{-1}) \qquad (2.36)$$

$$= \vartheta_0^{(r)} - \arccos[1 - \alpha^2 G^{-2} r^{-2}]^{1/2} \qquad (2.37)$$

Changing the integration constant by $\pi/2$ we can write instead

$$\vartheta^{(r)} = \vartheta_0^{(r)} + \arccos(\alpha G^{-1} r^{-1}) \qquad (2.38)$$

which, apart from the factor $\mathbf{G}^{-1}$, agrees with (I.A.21). Thus, since $Gr = R$, (2.38) results from equation (A.21) of I by again replacing "r" by **R**, as might have been anticipated. However, since **G** is a function of $\hat{\beta} = \omega\mathbf{r}/\mathbf{c}$, the geodesics are no longer straight lines and, moreover, depend on the angular velocity ω, except for the radial lines ($\alpha = 0$).

Note that our geodesics equations are still different from those given by Møller (1952) in terms of his nonmetrical coordinates.

As to the **C/D** ratio, we conclude from (2.24) that the circumference of a circle with radius $\mathbf{r}_i$ has the intrinsic measure

$$C_i = \kappa_i 2\pi R_i \qquad (2.39)$$

as in I. Since $\mathbf{D}_i = 2\mathbf{r}_i$ is the intrinsic measure of the diameter, we obtain

$$C_i/D_i = [\kappa_i G(\hat{\beta}_i)]\pi = [\kappa_i/g(\beta_i)]\pi \qquad (2.40)$$

Note that the use of $\mathbf{R}_i$ (instead of $\mathbf{r}_i$) in (2.39) is not only in line with the general replacement procedure ["r" $\to$ **R**] established in Section 2.1, but also in line with the fact that the elongation factor **g** applies only to measures in the radial direction.

Finally, we may compute the Riemann curvature tensor for the rotating disk surface. After a somewhat tedious calculation one obtains

$$R_{1212} = 0 \qquad R = 0 \tag{2.41}$$

where the indices refer to **r** and $\vartheta^{(r)}$, respectively. Thus the *rotating disk surface is everywhere uncurved*, independently of the choice of **g**. In connection with the two previous results this means that the choice of **g** affects only the "global geometry" on the disk [geodesics and **C/D** ratio] which has no definite physical meaning since a global geometry in the mathematical sense does not exist in a rotating frame as shown in I. [Note that $\vartheta^{(r)}$ is a *local* variable; if we had used instead the *global* nonmetrical variable ϑ defined by (2.24) we would have obtained a *nonvanishing* curvature tensor (cf. Appendix).]

2.6. Timelike Geodesics: The Experimental Determination of g. In contrast to the spacelike geodesics considered above, timelike geodesics have a definite physical meaning since they are the world lines of free test particles.

As an instance we consider the path in Σ^ω of a free particle moving with respect to Σ^0 according to the equations

$$\text{(i) } Z = 0 \qquad \text{(ii) } \theta = 0 \qquad \text{(iii) } R = VT \tag{2.42}$$

Its motion with respect to Σ^ω is then given by

$$\text{(i) } z = 0 \qquad \text{(ii) } \vartheta^{(r)} + \omega t^{(r)} = 0 \qquad \text{(iii) } rG = V\kappa_R^{-1}t^{(r)} \tag{2.43}$$

This yields the track equation

$$\vartheta^{(r)} = -\kappa_R G\omega r/V \tag{2.44}$$

and

$$\text{(i) } \dot{\vartheta}^{(r)} = -\omega \qquad \text{(ii) } \dot{r} = V(G\kappa_R)^{-1}[1 - \kappa_R^2\beta_R^2]^{-1}[1 + G^{-1}G'\beta_r]^{-1} \tag{2.45}$$

Thus, in principle it is sufficient to observe the track of a free particle in Σ^ω in order to determine the function **G** and hence **g**.

3. CONCLUDING REMARKS

In conclusion it should be emphasized that the theory of rotating frames here presented is a genuine extension of standard relativity, although its essential features are uniquely determined by the theoretical constraints listed in I. Hence it is not surprising that the function **g**, which determines the degree of geometrical anisotropy in the rotating plane, remains theoretically undetermined.

There is of course another kind of anisotropy in a rotating frame that is present also in the prerelativistic theory, viz., the inequivalence of opposite tangential directions. This inequivalence simply results from the fact that the angular velocity is an antisymmetric tensor and the angular speed ω not a scalar but a pseudoscalar. The questions connected with a relativistically correct or at least acceptable treatment of this *kinematic anisotropy* will be studied in a forthcoming paper.

APPENDIX: RIEMANN CURVATURE IN THE ROTATING PLANE

TABLE A1. In the following we give a comparative survey of the calculations concerning the Riemann curvature in the rotating plane. Note that the coordinates used in Strauss I, II are metrical, while those used in Møller (1952) are nonmetrical.

	Strauss I	Strauss II	Møller (1952)
$d\sigma^2$	$dr^2 + r^2\, d\vartheta^{(r)2}$	$[rG]l^2\, dr^2 + (rG)^2\, d\vartheta^{(r)2}$	$dR^2 + (\kappa R)^2\, d\vartheta^2$
γ_{11}	1	$[rG]_r{}^2$	1
γ_{22}	r^2	$(rG)^2$	$(\kappa R)^2$
	$\Gamma_{i,kl} =_{df} \frac{1}{2}(\gamma_{ik,l} + \gamma_{il,k} - \gamma_{kl,i})$		
$\Gamma_{1,11}$	0	$[rG]l[rG]_{rr}$	0
$\Gamma_{1,12}$	0	0	0
$\Gamma_{1,22}$	$-r$	$-rG[rG]_r$	$-\kappa R[\kappa R]_R$
$\Gamma_{2,11}$	0	0	0
$\Gamma_{2,12}$	r	$rG[rG]_r$	$\kappa R[\kappa R]_R$
$\Gamma_{2,22}$	0	0	0
	$\Gamma^i_{kl} =_{df} g^{im}\Gamma_{m,kl}$		
Γ^1_{11}	0	$[rG]_r^{-1}[rG]_{rr}$	0
Γ^1_{22}	$-r$	$-[rG]_r^{-1}rG$	$-\kappa R[\kappa R]_R$
Γ^2_{12}	r^{-1}	$[rG]_r(rG)^{-1}$	$(\kappa R)^{-1}[\kappa R]_R$
	$R_{iklm} =_{df} \frac{1}{2}(g_{il,km} + g_{km,il} - g_{im,kl} - g_{kl,im}) + \Gamma^s_{il}\Gamma_{s,km} - \Gamma^s_{il}\Gamma_{s,kl}$ $R_{1212} = \frac{1}{2}g_{22,11} + \Gamma^1_{11}\Gamma_{1,22} - \Gamma^2_{12}\Gamma_{2,12}$		
R	0	0	$6\kappa^4\beta^2 R^{-2}$
R_{1212}	0	0	$3\kappa^6\beta^2$

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